



SFI Working Paper Series

No. SFIWP0050

Synthetic CDO Pricing: The Perspective of Risk Integration

Conghui Hua Xun Zhang Qiuming Gao

14/05/2015

Shanghai Finance Institute (SFI) is a non-government and non-profit organization engaged in academic activities, supporting the development of Shanghai international financial center. SFI is operated by China Finance 40 Forum and has strategical cooperation with the Government of Huangpu District, Shanghai.

Copyright © 2015 by Shanghai Finance Institute. All rights reserved. No part of this working paper may be reproduced or utilized in any form or by any means, without permission from the SFI.

Synthetic CDO Pricing: The Perspective of Risk Integration

Conghui Hu^a Xun Zhang^{b,c,*} Qiuming Gao^d

Abstract

The underlying asset pool of Collateral Debt Obligations (CDOs) simultaneously encompasses credit risk and market risk. However, the standard CDO pricing model not only underestimates the risk to the asset pool due to a poor description of the correlation structure among obligors, but is also incapable of reflecting the impacts of interdependent markets, credit risks and systematic sudden shocks on the asset pool. This paper studies the joint impact of interrelated market and credit risk factors on the key inputs of CDO pricing (default probability, default correlation and default loss rate) under the framework of factor copula CDO pricing model and constructs a risk-integrated model for CDO pricing. In addition, we extend the static integrated model to a dynamic version by allowing the risk factors driven by the copula-GARCH process. The simulation results show that, compared with an integrated model, the premium of senior tranches is significantly lower under the standard model. Such difference is mainly due to different assumptions of the distributions of risk-driving factor.

Keywords: *Collateral Debt Obligation (CDO); Risk Integration; Factor Model; Copula*

JEL: *G12, G13, G19*

This paper is forthcoming at *Applied Economics*.

^a Business School, University of International Business and Economics, Beijing, China

^b Shanghai Finance Institute, Shanghai, China

^c China Center for Economic Research (CCER), National School of Development, Peking University, Beijing, China

^d Institute of Economics, Chinese Academy of Social Sciences, Beijing, China

* Corresponding author. Tel.: +86 15201468521. Email address: zhangxun@pku.edu.cn.

1. Introduction

Credit derivatives have played an active role in credit risk management around the world, and Collateral Debt Obligations (CDOs) have been one of the most rapidly developed credit derivatives. While the development of the global CDO market has been in full swing, the full-blown subprime crisis in the United States that began in August 2007 has cast a shadow over such development. The problem, which stemmed from the subprime mortgage market, quickly spread to global financial markets due to the credit's highly leveraged nature, leading a high number of mortgage lenders into bankruptcy and many world-renowned commercial banks, investment banks, and hedge funds into liquidity difficulties. In fact, the invalidity of the CDO pricing models was one of the important factors underlying the crisis. Therefore, it is vital to rethink the pricing theory of credit derivatives in ways other than attaching importance to financial regulations.

CDOs have two different structures--cash CDOs and synthetic CDOs. The underlying assets of cash CDOs can be seen as a series of default swaps in which the default probabilities of the lenders can be attained from the price of the credit default swap (CDS) markets, while those of the latter have poor liquidity (loans and bonds for instance). Therefore, we are not able to obtain default probabilities of synthetic CDOs from market price. In fact, pricing synthetic CDOs is always executed by rating agencies. However, as we will address below, standard synthetic CDO pricing not only underestimates the risk to the asset pool due to a poor description of the correlation structure among obligors, but is also incapable of reflecting the impacts of interdependent markets, credit risk and systematic sudden shocks on the asset pool. The pricing of synthetic CDOs is therefore biased. In this paper, we provide a risk-integrated framework for synthetic CDO pricing.

Although CDOs --with good attributes of asset securitization and credit derivatives-- can effectively transfer and disperse credit risks to meet the needs of different investors, CDOs carry multiple risks. The CDO asset pool is faced not only with the credit risks of debtors, but also market risks such as random changes of interest rates and asset prices. These market risks will have a significant impact on default probability, default correlation, and long-term discounted cash flow. Therefore, the risk premium of the CDO tranches should compensate for such market and credit risks.

There is a complicated interaction between market risk and credit risk, and the development of CDOs has established stronger internal mechanisms between these two risks. In the financial crisis in the U.S., one of the most important reasons for the accumulation of subprime mortgages was low interest rates over a long period. When interest rates began to increase steadily and real estate prices declined sharply,

the solvency of the debtor changed greatly, and the subprime mortgage crisis exploded. The crisis spread to other capital markets through CDOs, eventually resulting in extremely scarce market liquidity. This experience shows that CDOs and other credit derivatives are not only the link between market risk and credit risk, but also the bridge for the transferring of risk between different markets, such as from the real estate market and financial institutions to capital markets. Such interaction embodies a systemic feature of the risk. Particularly, superposition effects often leave the debtor with a sudden systematic risk shock. Therefore, we have to consider the interaction between risks when we assess the expected loss in the CDO asset pool.

The normal copula-factor model, the current mainstream CDO pricing model, not only underestimates the risk to the asset pool due to a poor description of the correlation structure among obligors, but is also incapable of reflecting the impacts of interdependent markets, credit risks and systematic sudden shocks on the asset pool. Therefore, this paper studies the joint impact of interrelated market and credit risk factors on the key inputs of CDO pricing (default probability, default correlation and default loss rate) under the framework of the factor copula CDO pricing model and constructs a risk integrated model for CDO pricing.

The model has the following characteristics: First, the factors of the market risk are introduced into the CDO pricing model by considering the impact of the stochastic risk-free rate on cash flow discounting and the influence of market risk driving factors on the debtors' credit status (including default probability and recovery). Second, the risk driving factor has a heavy tailed distribution so that it can describe the loss distribution caused by the debtors' default correlation structure. Third, market risk factors and credit risk factors have lower tail dependency that can measure the losses in the asset pool with the superposition of effects produced by the interaction between the risks. Fourth, a random recovery rate model is constructed in which the recovery rate depends on the debtors' systematic risk factors thereby influencing the credit status distinctly. Finally, default probability, the recovery rate and the risk-free rate are influenced by the same risk driving factor. On the one hand, these key parameters are mutually related. On the other hand, given the common risk factor, default probability, the recovery rate and the risk-free rate are independent of each other. In addition, in order to measure the impact from the change of the debtors' credit quality on CDO tranche pricing, we extend the static integrated model to a dynamic version by allowing the risk factors to be driven by the copula-GARCH process.

The rest of the paper proceeds as follows. Section 2 reviews the literature on CDO pricing in the framework of factor copula model. Section 3 sets forth a risk-integrated model for CDO pricing and extends the static-integrated model to a

dynamic version. We also discuss the implementation approach for the model. Section 4 details the method for obtaining the required model parameters from quotations. Further, the section illustrates the application of the integrated-risk model for synthetic CDO pricing by means of numerical examples. Section 5 concludes.

2. Literature Review

The literature on CDO pricing models can be divided into two strands according to the model design method: the top-down approach and bottom-up approach. The top-down approach models the change process of the CDO asset pool's total loss in accordance with the underlying asset's credit status. The bottom-up method is similar to the model for measuring portfolio credit risks, and includes the structure model, the intensity model and the factor copula model.

Copula model has much advantage over other models mentioned above. According to Sklar's Theorem, any multivariate joint distribution can be written in terms of univariate marginal distribution functions and a copula which describes the dependence structure between the variables. In this paper, we base on the dependence between market risk and credit risk to construct risk-integrated model for CDO pricing. Particularly, the correlation becomes more pronounced at the lower tail. Therefore, the copula function allows us to freely describe the dependence between random variables and therefore is suitable for our model setup. In fact, the normal factor copula model was initially set forth by Li (2000), and became the current mainstream model for CDO pricing.

The factor copula model measures the default correlation of the debtors using the copula function and the common factor. Semi-analytical expressions can be obtained using the factor copula model and thus it lowers the model risk. However, normal factor copula model faced many challenges. First, it is incapable of reflecting the impact of dynamic changes in credit quality on the asset pool. Second, it fails to fit the fat-tail character of the loss distribution, so the price based on the model is quite different from the CDO tranche's quotation. Thus, two ideas have been suggested to improve the standard model. The first uses other copula functions to describe the correlation between the debtors, such as the Student t copula function (Laurent and Gregory, 2005; Schloegl and O'Kane, 2005; Schönbucher and Schubert, 2001) and implied copula function (Hull and White, 2006). In such models, default probabilities of the debtors are obtained from the market quotation so correlation between the debtors is reflected. The second idea involves replacing the normal factor model with a fat-tail distribution function, such as the double t distribution (Hull and White, 2004), NIG distribution (Kalemanova et al., 2007), variance gamma distribution (Moosbrucker, 2006), heavy-tailed single-factor copula models (Wang et al., 2009), and the generalized hyperbolic distribution (Eberlein et al., 2008) (whose

special forms are NIG and variance gamma distribution, etc.). According to Andersen and Sidenius (2004), the correlation between the debtors is random and closely related to the business cycle. Therefore, Andersen and Sidenius (2004) extend the normal model to the random correlation coefficient model (RFL), and study the effect of a random recovery rate on CDO tranche pricing. Burtschell et al. (2009) compare the efficiencies of the factor models such as the normal copula model, the t-copula model, the Clayton copula model, the double t copula model, the M-O copula model and the two-state RFL model, and point out that the double t copula model and two-state RFL model better fit the synthetic CDO market quotation and the implied “correlation smile” curve. Burtschell et al. (2009) also explain the relationship between CDO tranche prices and the default correlation using the random order theory.

However, the copula models above only calculate the future expected losses over a fixed time and thus are static models. As such, they are not able to reflect the changes in the CDO asset loss over time. Although the Archimedean copula model has been established to extend the model to a dynamic version (Totouom and Armstrong, 2005; Totouom and Armstrong, 2007) and is able to cover the related structure of variables, the assumptions of homogenous and symmetric portfolios limit its practical application. Lamb et al. (2008) introduce an autoregressive process of the common factors and extend the normal copula model with a single factor to a dynamic situation, and the dynamic process of loss rate is achieved under the condition of a large sample. However, the model is not able to cope with a multiple factors model.

This paper studies the joint impact of interrelated market and credit risk factors on the key inputs of CDO pricing (default probability, default correlation and default loss rate) under the framework of the factor copula CDO pricing model and constructs a risk-integrated model for CDO pricing. Our contributions are mainly in two aspects: First, we introduce the Clayton copula function to describe tail-correlated market and credit factor. Second, we consider joint impact of interrelated market and credit risk factors on CDO pricing and construct a risk-integrated model for CDO pricing. This has more implications to the U.S. financial crisis than existing literature. Finally, existing CDO pricing literature, if not following static models, are with simple assumptions or single factor, this paper extends the static risk-integrated two-factor model to a dynamic version by allowing the risk factors driven by the copula-GARCH process.

3. Pricing Model for Synthetic CDOs

The prices of CDO tranches are actually the rates given to the protection seller. Therefore, we can build up pricing models by determining the parameters of each

CDO tranche. First, the expected loss of each tranche from the loss distribution of the asset pool is determined to achieve the default probability in certain timeframes. Second, the loss distribution of the asset pool is determined by the default probability, default loss rate and asset correlation. Third, we determine the joint default probability by looking at the differences between various CDO pricing models, which mainly involve the correlation structure determined by the common risk factors. Using the default loss rate and the distribution of common factors, the loss distribution of the asset pool at various times becomes obvious, See Figure 1.

[Insert Figure 1 approximately here]

3.1 Integrated Model

This paper studies the joint impact of interrelated market and credit risk factors on the key inputs of CDO pricing such as default probability, default correlation and default loss rate under the framework of the factor copula CDO pricing model and constructs a risk integrated model for CDO pricing. The market risk factors include the interest rate in different terms, the exchange rate, the stock index and real estate price, etc. The credit risk factors contain industry indexes, pledge prices and macroeconomic factors such as GDP, interest rate, inflation rate, unemployment rate for example. Due to the common risk driving factor, market risk and credit risk are correlated. In addition, the default correlation between different debtors can be derived from the common risk driving factor. Therefore, we can construct the integrated model from the common risk driving factor. On one hand, we can quantify the correlation between market risk and credit risk. On the other hand, we can also understand what systematically impacts the debtor's credit status and its correlation. Considering the large number of risk-driven factors, we first refine the risk factors to select the representative ones. In this paper, we choose the one-year Libor as a representation of market risk factors, and common macro-economic factors are obtained from the industry index.

3.1.1 Conditional Default Probability

There are n different debtors in the asset pool. X_i is i th debtor's credit status. Usually this is indicated by the standardized return on assets. The factor model can be built as:

$$X_i = \rho_c X_c + \rho_r X_r + \sqrt{1 - \rho_c^2 - \rho_r^2} \varepsilon_i \quad (1)$$

Where $\rho_c^2 + \rho_r^2 < 1$, X_c and X_r are the credit risk factor and market risk factor

respectively, ε_i is the specific risk factor for the i th debtor. $\rho_c(> 0)$ is the sensitivity coefficient of macro-economic factors. If the macro-economic factor has a higher value, the return on assets will be higher, and the debtor will be less likely to default. $\rho_r(< 0)$ is the sensitivity coefficient of interest rate. The borrowing cost of debtors will increase when the interest rate increases, resulting in the deterioration of the debtor's credit situation. Therefore, the poorer the macroeconomic condition and the higher the interest rate, the higher the risk of default.

In addition, we assume that ε_i is independent, and that it is independent from the common risk factors X_c and X_r . Furthermore, it follows standard t distribution with the degrees of freedom v_3 :

$$\varepsilon_i \sim t_{v_3}$$

The two-factor model set-up is actually a special case of the multiple factor setting in Hull and White (2004). By only considering two factors, we can more deeply explore the relationship between market risk and credit risk. As we have pointed out, the real situation is that complicated interaction exists between market risk and credit risk and has contributed significantly to the U.S. financial crisis. Therefore, we incorporate their relationship into the integrated model.

The marginal distributions of X_c and X_r follow the standard t distribution with the degrees of freedom v_1 and v_2 respectively. And they have the Clayton copula function with the same parameter θ , namely:

$$X_c, X_r \sim C_{cl}(u_1, u_2), u_1 = t_{v_1}(X_c), u_2 = t_{v_2}(X_r) \quad (2)$$

In fact, the Clayton copula function can describe the left tail correlation between the macro-economic factor X_c and the interest rate $-X_r$ (Charpentier and Juri, 2006).

The distribution of X_i , $H_i(X_i)$ is determined by the common factors: X_c , and X_r , as well as specific risk factors ε_i . When the common factors have a heavy tail, X_i is more likely to reach extreme values at the same time; thus the possibility of simultaneous assets default increases. When the specific risk factor has a thick tail, extreme values will occur on assets, and lower levels of loss are likely. The correlation between X_c and X_r in the integration model further increases the probability of simultaneous defaults.

$H_i(x_i)$ can be achieved by numerical simulation, and the default probability of the debtor is:

$$P_t^{i|X_c, X_r} = t_{v_3} \left[\frac{H_i^{-1}(F_i(t)) - \rho_c X_c - \rho_r X_r}{\sqrt{1 - \rho_c^2 - \rho_r^2}} \right] \quad (3)$$

3.1.2 Recovery Rate

In the standard CDO pricing model, the recovery rate is usually assumed to be constant. This is inconsistent with reality, as Altman et al. (2005) has shown that the recovery rate and default probability have a negative correlation through empirical investigation. Therefore, we assume that recovery rate is random and with a negative correlation with the asset pool's loss. Particularly, we set R_i as the recovery rate, which follows beta distribution. The use of the beta distribution to model a random recovery rate is simply following related literature (e.g. Gupton and Stein, 2002; Anderson and Sidenius, 2004). The first two moments of the beta distribution is approximated by the historical statistical data of recovery rates, see Eq. (4):

$$\begin{cases} R_i = F(\mu_i + \sigma_i \delta_i) \\ \delta_i = \alpha_i X_c + \beta_i X_r + \sqrt{1 - \alpha_i^2 - \beta_i^2} \eta_i \end{cases} \quad (4)$$

F is the cumulative distribution function of the beta distribution. μ_i, σ_i are the mean and standard deviation. $\alpha_i \in R^+, \beta_i \in R^-, \alpha_i^2 + \beta_i^2 < 1$. η_i follows standard normal distribution. Thus, the recovery rate is positively correlated to credit status δ_i , which is positively correlated to the macroeconomic factor X_c and negatively correlated to the interest rate factor X_r . According to Eq. (3), holding other factors constant, recovery rate is thus negatively correlated to the default probability.

3.1.3 Risk-free Rate

We assume that stochastic risk-free rate evolves as an Ornstein-Uhlenbeck process:

$$dr(t) = \kappa[\theta - r(t)]dt + \sigma dW(t), r(0) = r_0 \quad (5)$$

where $r_0, \kappa, \theta, \sigma$ are positive constants proposed by Vasicek (1977). $r(t)$ is normally distributed with mean $\theta + [r(0) - \theta]e^{-\kappa t}$ and volatility $\sqrt{\frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})}$.

The closed form solution for this stochastic differential equation is:

$$r(t) = \theta + [r(0) - \theta]e^{-\kappa t} + \sqrt{\frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})}W(t) \quad (6)$$

3.1.4 Total Loss Distribution Function

The recovery rate in the integrated model is stochastic. Thus the fast Fourier Transform method that assumes constant loss of units is not able to accurately calculate the distribution of the total loss in the integrated model.

Instead, we use the ‘‘probability bucketing’’ approach proposed by Hull and

White (2004) to implement the integrated model.¹ We divide the total losses by time T into the following intervals $\{b_1, b_2\} \dots \{b_{k-1}, b_k\} \dots \{b_{K-1}, b_K\}, \{b_K, \infty\}$. Our objective is to estimate the probability that the total loss lies in the k th bucket. Let p_k be the probability that the loss lies in the k th bucket by time T . Let A_k be the mean loss in the k th bucket.

We can obtain the value of p_k and A_k through an iterative procedure. First, we assume that there are no assets in the asset pool, so $p_1 = 1, p_k = 0$ for $k > 1$. We also have $A_1 = 0, A_k = 0.5(b_k + b_{k+1})$ for $2 \leq k \leq K - 1$ and $A_K = b_K$. Suppose that we have calculated the p_k and A_k when the first $j-1$ debt instruments are considered and that the loss given default from the j th debt instrument is L_j and the probability of default is α_j . Define $u(k)$ as the bucket containing $A_k + L_j$ for $1 \leq k \leq K$. When $u(k) > k$, the updating equation system is:

$$\begin{cases} p_k = p_k^* - p_k^* \alpha_j \\ p_{u(k)} = p_{u(k)}^* + p_k^* \alpha_j \\ A_k = A_k^* \\ A_{u(k)} = \frac{p_{u(k)}^* A_{u(k)}^* + p_k^* \alpha_j (A_k^* + L_j)}{p_{u(k)}^* + p_k^* \alpha_j} \end{cases} \quad (7)$$

where $p_k^*, p_{u(k)}^*, A_k^*$ are the values of $p_k, p_{u(k)}, A_k$ before the k th asset is considered. When $u(k) = k$, the updating equations are:

$$\begin{cases} p_k = p_k^* \\ A_k = A_k^* + L_j \alpha_j \end{cases} \quad (8)$$

3.1.5 Pricing Equation of Synthetic CDO Tranche

Synthetic CDO pricing involves two main bodies: the issuer, i.e., the buyer of the protection, and the investor, i.e., the seller of the protection. Investors receive fixed cash flow from the issuers regularly. This is the return side. When the underlying assets of a CDO are affected by the loss of the asset pool, investors are required to compensate the buyers for the loss correspondingly. This is the loss side. Synthetic CDO tranche pricing is used to determine a reasonable rate of return to equalize the discounted value of the cash flows for the two sides with a risk-neutral probability. Synthetic CDOs' cash flow depends on the cumulative loss distribution of the underlying assets. Therefore, we can obtain the CDO tranche price according to the calculated loss distribution of the asset pool.

1) Loss Side

For example, when the portion $[a, b]$ of the underlying asset pool suffers losses,

¹ For detailed steps of the "probability bucketing" approach, please refer to Hull and White (2004).

the seller of the protection will compensate the buyer. $L_t^{[a,b]}$ is the accumulated loss:

$$L_t^{[a,b]} = [L(t) - a]^+ - [L(t) - b]^+ \quad (9)$$

$E^Q [L_t^{[a,b]}]$ is the expected loss for the tranche $[a, b]$ with the risk-neutral probability at time t . Then,

$$\begin{aligned} E^Q [L_t^{[a,b]}] &= (b - a)Q[L(t) > b] + \int_a^b (x - a)dF_L^Q(x) \\ &= (b - a)Q[L(t) > b] + \sum_{k=a}^b (k - a)Q[L(t) = k] \end{aligned} \quad (10)$$

$L_t^{[a,b]}$ is a jump process, so the default payment is a augmentation of $L_t^{[a,b]}$. In other words, when $L_t^{[a,b]}$ jumps, the payment is $L_{t+}^{[a,b]} - L_t^{[a,b]}$. Define Stieltjes integration of $L_t^{[a,b]}$, and then the expected discounted value of loss for the tranche $[a, b]$ is:

$$E[DL] = E^Q \left[\int_0^T EB(0, t) dL_t^{[a,b]} \right] \quad (11)$$

Where

$$EB(0, t) = E \left[\exp \left\{ - \int_0^t r(s) ds \right\} \right] \quad (12)$$

2) Return Side

The seller of the protection can get regular periodic premiums with the fixed rate s by the buyer. The principal is the value of tranche $[a, b]$ at the expiration date, and $\Delta_i = t_i - t_{i-1}$ is the payment frequency. To simplify the calculation, we ignore the accrued interest when default occurs between two payment days, then the expected discounted value of revenue for the tranche $[a, b]$ is:

$$E[PL] = s \sum_{i=1}^I EB(0, t_i) \Delta_i \left(b - a - E^Q [L_T^{[a,b]}] \right) \quad (13)$$

Therefore, the fair price of the tranche $[a, b]$ is:

$$s = \frac{E[DL]}{E[PL]} = \frac{EB(0, T) E^Q [L_T^{[a,b]}] + \int_0^T E[r(t)] E^Q [L_t^{[a,b]}] EB(0, t) dt}{\sum_{i=1}^I EB(0, t_i) \Delta_i (b - a - E^Q [L_T^{[a,b]}])} \quad (14)$$

3.2 Dynamic Integrated Model

Research has been conducted on the dynamic CDO pricing models. In the top-down approach, we model the change process of the CDO asset pool's total loss in accordance with the underlying asset's credit status. In the bottom-up approach, which we simulate the dynamic process of assets loss, and then solve the cumulative loss distribution of the asset pool. However, research is mostly based on structured or simplified models. Rare studies use the factor copula model to analyze the dynamic process of the loss. The factor model's biggest flaw is that the entire loss on the asset pool is only dependent on the simulation values of the factors driving risk at the beginning time. This means that the credit circumstance will not change for five to ten years, which obviously does not correspond to reality. Therefore, we extend the model by introducing the timing characteristics of the risk driving factors. In this way, we can examine the impact of the debtor's credit quality changes, i.e., the timing characteristics of default probabilities and recovery rate, on the CDO tranche pricing.

3.2.1 Model Construction

In order to construct the dynamic model, we extend Equation (1) and (4):

$$X_{it} = \rho_c X_{ct} + \rho_r X_{rt} + \sqrt{1 - \rho_c^2 - \rho_r^2} \varepsilon_{it} \quad (15)$$

$$\left\{ \begin{array}{l} R_{it} = F(\mu_i + \sigma_i \delta_{it}) \\ \delta_{it} = \alpha_i X_{ct} + \beta_i X_{rt} + \sqrt{1 - \alpha_i^2 - \beta_i^2} \eta_{it} \end{array} \right. \quad (16)$$

The parameter restrictions are the same as the static model. We capture the correlation and timing characteristics of the risk driving factors X_c and X_r using the AR-copula-GARCH model. Specifically, $X_{jt}, j = c, r$ follows the GARCH process:

$$\left\{ \begin{array}{l} X_{jt} = \mu_j + \sum_{i=1}^p \phi_{ij} X_{j,t-i} + \varepsilon_{jt} \\ \varepsilon_{jt} = \sqrt{h_{jt}} u_{jt} \\ h_{jt} = w_j + c_j \varepsilon_{j,t-1}^2 + d_j h_{j,t-1} \\ (u_{ct}, u_{rt}) | I_{t-1} \sim C_{cl}(t_{v_1}(u_{ct}), t_{v_2}(u_{rt})) \end{array} \right. \quad (17)$$

In other words, we model the macroeconomic factor and interest rate factor with a p th-order autoregressive process, and assume that their marginal distributions follow the standard t distribution of free degree of v_1 and v_2 respectively.

3.2.2 Model Implementation

Monte Carlo simulation is used to implement the model (15)-(17). The fixed payment times are $t_0, t_1 \cdots t_k \cdots t_K$.

Step 1: Estimate the parameters using market quotations;

Step 2: Forecast $h_{jt_k}, j = c, r, k = 1 \cdots K$ based on the AR-copula-Garch model;

Step 3: Simulate u_1, u_2 on 0 to 1 using copula function at t_k for $k = 1 \cdots K$, which represents the correlation structure of $C_{cl}(u_1, u_2)$;

Step 4: Get the path for $X_{jt_k}, j = c, r, k = 1 \cdots K$ from step 2 and 3;

Step 5: Assume that default is in the absorbing state. Let function $F_i(t_k)$ be the probability of the first default of debtor i between t_{k-1} and t_k ;

Step 6: When $F_i(t_k)$ is known, $H_i^{-1}(F_i(t_k))$ is the threshold value of the debtor default at $[t_{k-1}, t_k]$. $H_i(\cdot)$ is the distribution function of X_i ;

Step 7: d_k is set as the number of default assets for the period of $[t_{k-1}, t_k]$, and the default loss is $l_k = \sum_{i=1}^{d_k} N_i(1 - R_i)$. $L_k = l_k + L_{k-1}$ is the cumulative default losses till t_k ;

Step8: Let

$$P_k^{[a,b]} = \begin{cases} b - a, & \text{if } L_k < a \\ b - L_k, & \text{if } a < L_k < b \\ 0, & \text{if } L_k > b \end{cases} \quad (18)$$

Step9: The discounted value of the return side of tranche $[a, b]$ is:

$$A = \sum_{k=1}^K (t_k - t_{k-1}) P_k^{[a,b]} EB(0, t_k) \quad (19)$$

The discounted value of the loss side is:

$$B = \sum_{k=1}^K [P_{k-1}^{[a,b]} - P_k^{[a,b]}] EB(0, t_k) \quad (20)$$

Step10: The price of tranche $[a, b]$ is $s = \frac{\hat{B}}{\hat{A}}$. \hat{A} and \hat{B} are the expected values from the Monte Carlo simulation.

From the above steps of the implementation of the dynamic model, we can see that the first default probability of debtor i on $[t_{k-1}, t_k]$ is the key to the model.

Usually, the debtor i 's risk-neutral default probability can be obtained from a bond price with higher liquidity or the spread of credit default swap. Here we introduced a calculation method for $F_i(t_k)$ provided by Hull and White (2001).

$q(t)\Delta t$ is the default probability between t to $t + \Delta t$ observed at the time 0. $h(t)\Delta t$ is the default probability between t to $t + \Delta t$ observed at the time t , which assumes that there is no default in $[0, t]$. The relationship between $q(t)$ and $h(t)$ is:

$$q(t) = h(t) \exp \left\{ - \int_0^t h(s) ds \right\} \quad (21)$$

The $q_i(t_k)$ represents the $F_i(t_k)$ above. Here we abbreviate $q_i(t_k)$ as q_{ik} . We also assume that $q(t)$ is constant as q_k between t_{k-1} and t_k . The price of the bond issued by debtor i at t_0 is B_i . G_i is the price of risk-free bond at t_0 . Assume that we can get k bonds with high liquidity and identical credit status, and the expiration date is t_k , $k = 1 \cdots K$ respectively. $C_i(t)$ represents the expected loss for the debtor i at time t . The discounted rate is $v(t)$. The present value of default for the i th bond at t_k is $\beta_{ik} = \int_{t_{k-1}}^{t_k} v(t) C_i(t) dt$. And, $G_i - B_i = \sum_{k=1}^i P_{ik} \beta_{ik}$. Therefore, we have

$$P_{ii} = \frac{G_i - B_i - \sum_{k=1}^{i-1} P_{ik} \beta_{ik}}{\beta_{ii}}, i = 1 \cdots k \quad (22)$$

This represents the first default probability for the i th bond at $[t_{i-1}, t_i]$.

4. Examples for the Synthetic CDO Pricing in the Integrated Model

4.1 Parameter Estimation

The parameters of the synthetic CDO pricing model and dynamic integration model are divided into the following categories: the debtor's default probability in underlying asset pool, factor-loading coefficients in the factor model, the parameters of the factor's distribution and the parameters in the recovery model, and the parameters in the dynamic copula-GARCH model. We will give details on how to estimate these parameters from the available data.

4.1.1 Default Probability

In order to obtain the joint distribution function of default, we first need to obtain the debtor's default probability, that is, the marginal distribution of default.

Assume that default time τ_i follows exponential distribution $Q_i(t) = 1 - e^{-\lambda_i t}$. Default intensity λ_i can be estimated by the spreads of the identical CDS debtors. The relationship between the CDS spread and the default intensity are approximate as $\lambda_i \approx \frac{r_{CDS}^i}{1-R}$ (Duffie and Singleton, 2003).

4.1.2 Factor Model

The factor coefficient of the normal copula-factor model is the coefficient ρ between the debtors. We can get the implied coefficient from the quotation of standard tranches, similar to getting the implied volatility from the Black-Scholes pricing model. We can get the implied coefficient from the equity tranche, and then determine the prices of other tranches. Similarly, we can also get the parameters of the factor-loading coefficient through the quotation of the high-liquidity DJ iTraxx (CDX) index standard tranches.

However, more than one parameter requires estimation. Therefore, we use the least square method. The basic idea is to minimize the sum of the square of the difference between the quotation and the integrated model. This method promotes solving for the implicated coefficient. The method of solving for the implied coefficient ensures that the pricing of the model fits the quotation closely. The least square method offers a comprehensive consideration of the impact of model parameters of all the tranches' quotations and improves the fitness of the quotation on the whole.

4.1.3 Recovery Rate

The parameters of the beta distribution in the recovery rate model can be determined by the mean and variance of the historical recovery rate. The factor-loading coefficient can be obtained from the historical data of the recovery rate in (4) and (16) by using the least square method.

4.1.4 Copula-GARCH Model

The parameters in the copula-GARCH model can be estimated from the historical data of the macro-economic factors and interest rate factor by using the two-stage maximum likelihood estimation.

The first stage is to estimate the GARCH equations, i.e., Equation (17) of the macro-economic factors and the interest rate factor, and determine the degrees of the

freedom parameter ν in the marginal distribution of the t distribution.

The second stage is to estimate the Clayton copula function parameter θ using the maximum likelihood estimation method:

$$\begin{cases} u_1^j = t_{\nu_1}(u_{ct}^j) \\ u_2^j = t_{\nu_2}(u_{rt}^j) \\ \hat{\theta} = \text{Arg max}_{\theta} \sum_{j=1}^N \ln[c_{\theta}(u_1^j, u_2^j)] \end{cases} \quad (23)$$

where $c_{\theta}(\cdot)$ is the intensity function of the Clayton copula. In fact, the copula-GARCH model is more flexible than a normal multivariate GARCH copula in the parameters estimation.

4.2 Numerical Simulation

4.2.1 Key Parameters Comparison between the Standard and Integrated Model

We assume that X_c, X_r, ε_i are all independent from each other, and they follow the standard normal distribution. The risk-free interest rate and recovery rate are constants. Due to data limitations of the CDO tranches quotation, we will explain the differences of the key parameters of the standard model and integrated model through numerical examples.

Consider a 5-year synthetic CDO on a basket of 100 CDSs with different credit. There are three tranches according to the level of risk and return: equity, mezzanine and senior. They bear the losses in the CDO asset pool, 0%-5%, 5%-25% and 25%-100%, respectively. The intensity of the debtors λ_i is assumed to have uniform distribution on 0.0005 to 0.025, and the mean is 0.015. Table 1 contains the parameters in the standard and integrated models.

[Insert Table 1 approximately here]

The integrated model captures the correlation between the debtors through the heavy tail of the risk driving factors. Furthermore, in order to consider the impact of associated market risk and credit risk on the credit status of the debtor, we use the Clayton copula function with a right tail to describe the correlation structure of market risk and credit risk factors. Figure 2 shows the differences of the risk driving factors distribution between the integrated model and the standard model. Compared with the standard model, there is more risk driving factors in the integrated model with the heavy tail, especially the left tail. In this way, we can characterize the impact on the CDO tranches price when the market environment deteriorates and a

verity of debtors default at the same time. In addition, when the recovery rate is fixed, the total loss distribution will be solely determined by the distribution of the default probability. Therefore, the character of the total loss in the two models can be reflected by comparing the difference of the default probability distribution in the integrated model and the standard model.

[Insert Figure 2 approximately here]

Figure 3 depicts the relation between the total loss distribution and conditional default probability of each obligator. The results show that the default probability distribution is closer to the Y-axis (i.e. to the completely related condition) for the integrated model. It implies that for a given conditional default probability, the whole asset pool have a heightened likelihood to experience loss under the integrated model. This evidence suggests that the integrated model captures a high correlation between the debtors compared to the standard model.

[Insert Figure 3 approximately here]

When default probabilities and recovery rates are given, we can calculate the expected loss of the CDO tranches by the approach set forth in Section 3.2.2. The CDO prices can be obtained from the CDO pricing model in the Section 3.2.3.

Table 2 provides the fair yield of all the CDO tranches in the standard model and integrated model. Results in Table 2 show that the yields (in basis points) of equity and senior tranches are 8% and 142% higher, respectively in the integrated model than in the standard one. On the other hand, the premium of mezzanine tranche is 29% lower. Possible explanation is that the standard model underestimates the fat-tail loss.

We find some support for the underestimation. We consider the integrated models in two situations. The first is the integrated model with a constant risk-free interest rate, i.e., the integrated model CI. Another is the integration model with a constant recovery rate, i.e., the integrated model CR. The estimation results are shown in Table 2.

[Insert Table 2 approximately here]

We find that there is no obvious difference between integrated models with a constant risk-free rate. Thus the impact of a stochastic risk free rate on the discounted cash flow is incapable of explaining the difference between the standard model and the integrated model. The constant recovery rate does not influence the pricing of the equity tranche or the senior tranche. However, the yield of the senior

tranche is significantly reduced in the integrated model under the assumption of a constant recovery rate. Therefore, the impact of the random recovery rate is significant on the asset loss distribution.

However, the price of the senior tranche in the integrated model with a constant recovery rate is much higher than the one in the standard model. In other words, the characterization of the default correlation structure is the main reason for the difference between the models, and the recovery rate is also an important factor.

4.2.2 Changes in the Parameters of the Integrated Model

1) Degree of Freedom ν

The accurate characterization of the default correlation is the key to the integrated model. We describe the correlation between the debtors by the correlation coefficient ρ in the normal copula-factor model. In the integrated model, in addition to systemic risk loading factors ρ_c, ρ_r , the parameters of risk factors and the correlation of the systemic risk factors also influence the relation between the debtors greatly. Therefore, they all affect the pricing of the CDO tranches in the integrated model.

We further discuss the relationship of the CDO tranche price and the default correlation structure. Research has been conducted on the relationship between CDO tranche price and the correlation coefficient ρ . That research has concluded that if the correlation coefficient ρ becomes bigger, the price of an equity tranche will be lower and the senior tranche price will be higher. Here we discuss the relationship between the risk factor distribution parameter ν , systemic risk factors related structure parameters θ and the price of CDO tranches.

Table 3 shows the CDO tranche prices with different degrees of freedom correspondingly. $\nu(\nu > 2)$ is the parameter used to describe the distribution of tail correlation. The smaller the value, the greater the relevance of the distribution of the tail, and the heavier the tail is. As ν tends to infinity, the t -distribution converges to standard normal distribution, without tail correlation.

[Insert Table 3 approximately here]

As illustrated in Table 3, equity and senior tranches are more expensive as ν decreases. When ν decreases, the tail of risk factors becomes heavier. A heavier tail of specific risk factors also means that the default probability for a person increases; as there will be smaller losses, the price of the senior tranche increases.

2) Factor Correlation Parameter θ

Table 4 shows the different prices of CDO tranches under various θ , the parameters describing the related structure between the systemic risk factors. When θ increases the left tail correlation of the Clayton copula function is enhanced, and the likelihood of taking extreme values to the common risk factors increases. Thus the risk of default of a multiplicity of debtors increases at the same time. As illustrated in the Table 4, the senior tranche price will increase with θ .

[Insert Table 4 approximately here]

3) Systemic Risk Loading Factors ρ_c, ρ_r

Table 4 shows the different prices of CDO tranches under various systemic risk loading factors. The results shows that the premium of senior (equity) tranche increases (decreases) with the absolute correlation ρ , while the relation between the price of mezzanine tranche and coefficient ρ is uncertain. This is consistent with previous findings in the literature (e.g. Hull et al., 2005).

[Insert Table 5 approximately here]

4.3 Examples in the Dynamic Integrated Model

The probability of the first default for the debtor i between t_{k-1} and t_k is the key to the dynamic pricing model. We rate the credit of underlying assets into different levels. In Table 6, we show the density of default probability q_k of the BBB rating. Here we assume that the density of default probability is constant within one year. It is noteworthy that the frequency of CDO tranche payments is usually 0.25 years. Therefore, we need to get the simulate values of the risk factors from the copula-GARCH model based on one year.

[Insert Table 6 approximately here]

The dynamic model considers the impact of timing characteristics of the systemic risk driving factors on the CDO pricing. The financial time series often have timing and clustering features, which means that a large/small fluctuation usually follows another one. This character of volatility is greatly significant for capturing the dynamic behavior of the credit environment over time. A GARCH model can better characterize the volatility of financial time series, and thus can better describe the conditions for risk driving factors distribution. Therefore we can

correctly obtain the impact of the change of a debtor's credit quality on CDO pricing. A high regression coefficient in the GARCH model means that when a shock occurs, multi-periods of credit will be affected, which will influence the cumulative loss of the entire asset pool. Table 7 shows the CDO tranches pricing results for different regression coefficients in the dynamic integrated model. Obviously, as the coefficients are higher, the CDO tranche prices are higher.

[Insert Table 7 approximately here]

5. Conclusion and Further Research

CDOs are a financial instrument with multiple risks. Market risk and credit risk have complicated interaction, and the development of CDOs has established stronger internal mechanisms between the two risks. The CDO pricing model should be able to reflect the effect of the interaction of market risk and credit risk on losses in the CDO asset pool.

The integrated model in this paper not only describes the character of the fat-tail of loss distribution, but also reflects the systemic risk caused by the superposition of different risk levels. Meanwhile, default probability, recovery and risk-free interest rates are influenced by the common risk driving factors. The model not only makes the portrayal of pricing parameters and the relationship more objective, but it also facilitates calculation and implementation. In addition, the dynamic integrated model may reflect the impact of the change of the credit quality of the debtors' asset pool on the expected loss.

The simulation results show that compared to the standard model, the premium of the equity and senior tranches is higher while the premium of the mezzanine tranche is lower under the integrated model. The main reason for such differences is the different characterization of default correlation structures, especially the interaction between market risk and credit risk. In fact, market risk factors and credit risk factors have lower tail dependency that can measure the losses in the asset pool with the superposition of effects produced by the interaction between the risks. The recovery rate is also an important factor affecting CDO pricing. However, the risk-driving factors determine the debtor's default correlation structures, as well as the recovery rate. In some sense, CDO tranche pricing will ultimately depend on the risk-driving factors. Reducing degrees of freedom ν of t distribution and increasing the Clayton copula parameters θ have the same effect as increasing the correlation coefficient ρ . The result is that the higher the ρ in a normal copula-factor model, the higher the yield of the senior tranche and the lower the yield of the equity tranche. In the dynamic integrated model, the regression coefficient plays an important role in the pricing of synthetic CDOs.

Due to data limitation, we are not able to conduct empirical comparisons between the standard model and our integrated model, nor can we discuss the hedging performance of the integrated model with respect to the model developed by Frey and Backhaus (2010). The focus of this paper is to propose an alternative CDO pricing that emphasizes the impact of interdependent market and credit risk on the asset pool and explain how it works in numerical examples. Empirical comparisons in the spirit of Longstaff and Rajan (2008), and assessing the pricing and hedging performance using real data are both worth independent research in the future.

References

- [1] Altman, E. I., Brady, B., Resti, A., & Sironi, A. (2005). The Link between Default and Recovery Rates: Theory, Empirical Evidence, and Implications. *The Journal of Business*, 78(6), 2203-2228.
- [2] Andersen, L., & Sidenius, J. (2004). Extensions to the Gaussian Copula: Random Recovery and Random Factor Loadings. *Journal of Credit Risk* Volume, 1(1), 05.
- [3] Burtschell, X., Gregory, J., & Laurent, J. P. (2005). A Comparative Analysis of CDO Pricing Models. In ISFA Actuarial School and BNP Parisbas.
- [4] Charpentier, A. & Juri, A. (2006). Limiting Dependence Structures for Tail Events, with Application to Credit Derivatives. *Journal of Applied Probability*, 43, 213–246.
- [5] Duffie, D., and K. Singleton (2003), *Credit Risk*, Princeton, Princeton University Press.
- [6] Eberlein, E., Frey, R., & von Hammerstein, E. (2008). A. Advanced Credit Portfolio Modeling and CDO Pricing. Mathematics–Key Technology for the Future, 253.
- [7] Frey, R. and Backhaus, J. (2010). Dynamic Hedging of Synthetic CDO Tranches with Spread Risk and Default Contagion. *Journal of Economic Dynamics & Control*, 34, 710-724.
- [8] Gupton, G. M., Stein, R. M., Salaam, A., & Bren, D. (2002). LossCalc™: Model for predicting loss given default (LGD). Moody's KMV, New York.
- [9] Hull, J. C., & White, A. D. (2001). Valuing Credit Default Swaps II: Modeling Default Correlations. *The Journal of derivatives*, 8(3), 12-21.
- [10] Hull, J. C., & White, A. D. (2004). Valuation of a CDO and an n-th to Default CDS without Monte Carlo Simulation. *The Journal of Derivatives*, 12(2), 8-23.
- [11] Hull, J. C., Predescu, M., & White, A. (2005). The valuation of correlation-dependent credit derivatives using a structural model. Available at SSRN 686481.
- [12] Hull, J. C., & White, A. D. (2006). Valuing Credit Derivatives Using an Implied Copula Approach. *The Journal of Derivatives*, 14(2), 8-28.
- [13] Kalemanova, A., Schmid, B., & Werner, R. (2007). The Normal Inverse Gaussian Distribution for Synthetic CDO Pricing. *The Journal of Derivatives*, 14(3), 80-94.
- [14] Lamb, R., Perraudin, W., & Van Landschoot, A. (2008). Dynamic Pricing of Synthetic Collateralized Debt Obligations. Imperial College.
- [15] Laurent, J. P., & Gregory, J. (2005). Basket Default Swaps, CDOs and Factor Copulas. *Journal of Risk*, 7(4), 103-122.
- [16] Li, D. X. (2000). On Default Correlation: A Copula Function Approach. The

- Journal of Fixed Income*, 9(4), 43-54.
- [17] Longstaff, F. and Rajan, A. (2008). An Empirical Analysis of the Pricing of Collateralized Debt Obligations. *Journal of Finance*, 63, 529-563.
- [18] Moosbrucker, T. (2006). Pricing CDOs with Correlated Variance Gamma Distributions. *Journal of Fixed Income*, 12, 1-30.
- [19] Schloegl, L., & O’Kane, D. (2005). A Note on the Large Homogeneous Portfolio Approximation with the Student-t Copula. *Finance and Stochastics*, 9(4), 577-584.
- [20] Schönbucher, P., & Schubert, D. (2001). Copula-dependent Default Risk in Intensity Models. No. TIK-Report, 103.
- [21] Totouom, D., & Armstrong, M. (2005). Dynamic Copula Processes: A New Way of Modelling CDO Tranches. The 5th Annual Advances in Econometrics, Louisiana State University.
- [22] Totouom, D., & Armstrong, M. (2007). Dynamic Copula Processes: Comparison with Five 1-factor Models for Pricing CDOs. Working Paper, Cerna, Ecole des Mines de Paris.
- [23] Vasicek, O. (1977). An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, 5(2), 177-188.
- [24] Wang, D., Rachev, S. T., & Fabozzi, F. J. (2009). Pricing of Credit Default Index Swap Tranches with One-factor Heavy-tailed Copula Models. *Journal of Empirical Finance*, 16(2), 201-215.

Table 1**Parameters in the Standard Model and Integrated Model**

Parameters in the Synthetic CDO Model
<ol style="list-style-type: none">1. Number of underlying assets: 100; Face value: 1; Maturity: 5 years.2. Default intensity: 0.015; Payment frequency: 0.25 years.3. Equity tranche: 0%-5%; Mezzanine tranche: 5%-25%, Senior tranche: 25%-100%;
Parameters in the Standard Model
<ol style="list-style-type: none">1. $\rho_c = 0.2$; $\rho_r = -0.05$;2. Risk-free Rate $r = 6\%$;3. Recovery Rate $R = 40\%$.
Parameters in the Integrated Model
<ol style="list-style-type: none">1. $\rho_c = 0.2$; $\rho_r = -0.05$;2. $v_c = v_r = v_i = 4, i = 1 \dots 100$;3. θ(Parameter in the Clayton Copula Model) = 2;4. Recovery Rate Model Parameters: Mean=0.4; Standard Deviation=0.35; $a = 0.4$; $\beta = -0.25$;5. Risk-free Rate Parameters: $\theta = 0.06$; $r_0 = 6\%$; $k = 0.4$; $\sigma_r = 0.01$.

Table 2**CDO Tranche Prices in Different Models**

CDO Tranche (bps, per annum)	Standard Model	Integrated Model	Integrated Model CI	Integrated Model CR	Integrated Model SR
Equity Tranche (0%-5%)	1909	2062	2045	2039	2046
Mezzanine Tranche (5%-25%)	764	544	572	512	534
Senior Tranche (25%-100%)	43	104	100	86	79

Table 3**CDO Tranche prices for Different Parameters ν**

CDO Tranche (bps, per annum)	$\nu = 2.5$	$\nu = 3$	$\nu = 4$	$\nu = 5$
Equity Tranche (0%-5%)	2401	2112	2062	2042
Mezzanine Tranche (5%-25%)	503	535	544	562
Senior Tranche (25%-100%)	136	115	104	96

Table 4**CDO Tranche prices for Different Parameters θ**

CDO Tranche (bps, per annum)	$\theta = 0.1$	$\theta = 0.5$	$\theta = 2$	$\theta = 4$
Equity Tranche (0%-5%)	2123	2096	2062	2030
Mezzanine Tranche (5%-25%)	576	553	544	545
Senior Tranche (25%-100%)	80	95	104	121

Table 5**CDO Tranche prices for Different Parameters ρ_r, ρ_c**

CDO Tranche (bps, per annum)	$\rho_c = 0.1$ $\rho_r = -0.025$	$\rho_c = 0.2$ $\rho_r = -0.05$	$\rho_c = 0.3$ $\rho_r = -0.075$
Equity Tranche (0%-5%)	1905	2062	2532
Mezzanine Tranche (5%-25%)	556	544	565
Senior Tranche (25%-100%)	114	104	96

Table 6**Default Probability Intensity for Rating BBB**

Default Time (Year)	Default Probability Intensity
0~1	0.0219
1~2	0.0242
2~3	0.0264
3~4	0.0285
4~5	0.0305

Table 7**CDO Tranche Prices in the Integrated Model**

CDO Tranche (bps, per annum)	$\phi_{1c} = \phi_{1r} = 0.2$ $d_c = d_r = 0.6$	$\phi_{1c} = \phi_{1r} = 0.4$ $d_c = d_r = 0.8$
Equity Tranche (0%-5%)	1987	2362
Mezzanine Tranche (5%-25%)	602	531
Senior Tranche (25%-100%)	102	131

Note: 1. $c_c = 0.15$; $c_r = 0.1$;

2. $\mu_c = \mu_r = w_c = w_r = 0$;

3. $\sigma_c = 0.2$; $\sigma_r = 0.01$;

4. Other parameters are the same with the ones in the integrated model, see Table 1.

Figure 1
Main Steps of CDO Tranche Pricing

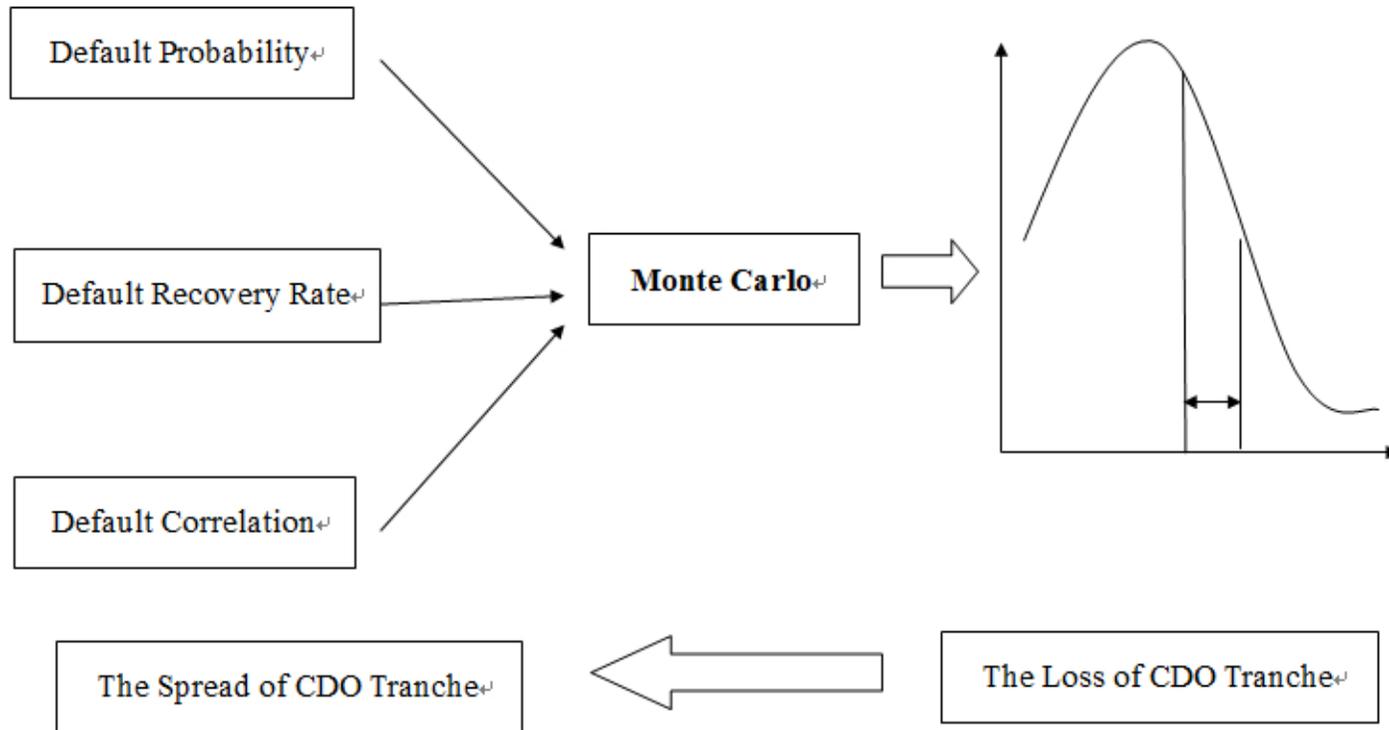


Figure 2
Simulated Distribution of the Risk Driving Factors

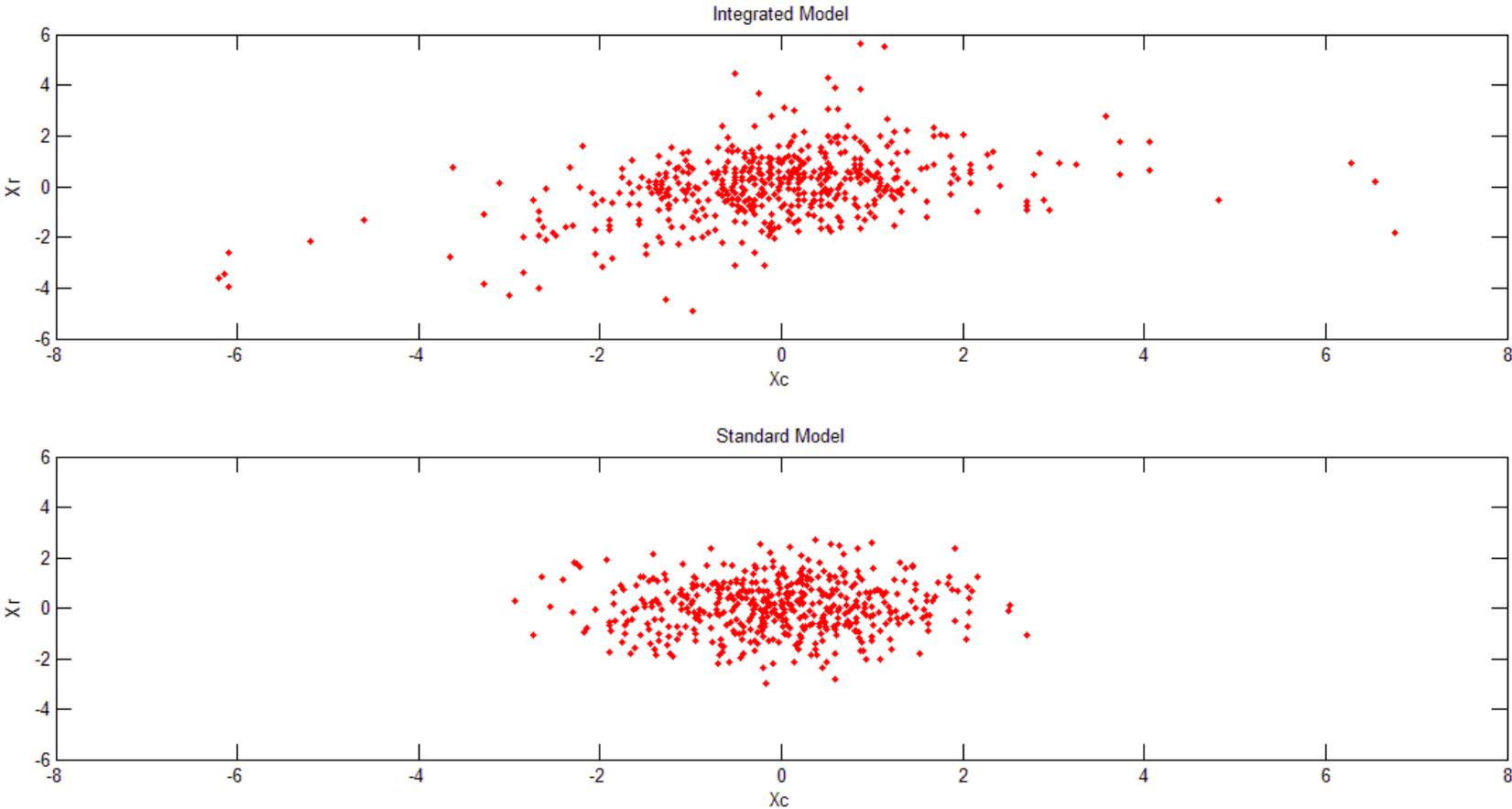


Figure 3
Default Probability Distribution

